Cross Product – Appendix A

ex. given $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$, determine $\vec{v} \times \vec{w}$

"cross"

recall: $\vec{e}_1 = i$

$\vec{e}_2 = j$ in $\mathbb{R}^3$

$\vec{e}_3 = k$

set up: $\begin{bmatrix} i & j & k \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ construct 3x3 matrix

recall 2x2 determinant: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A \Rightarrow \det A = ad - bc$

know A is invertible iff $\det A \neq 0$

alternate signs (3 terms): $+$ $-$ $+$

$\begin{bmatrix} i & j & k \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ $\begin{bmatrix} i & j & k \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ $\begin{bmatrix} i & j & k \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$

$\vec{v} \times \vec{w} = + \det \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix} i - \det \begin{bmatrix} 2 & 4 \\ 5 & 7 \end{bmatrix} j + \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} k$

$= (21 - 24)i - (14 - 20)j + (12 - 15)k$

$= -3\underbrace{i}_{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} - (-6)j - 3k \Rightarrow \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$

Section 6.1 Expand Def'n of Determinants

if $A = [\vec{u}\ \vec{v}\ \vec{w}]$ then $\boxed{\det A = \vec{u} \underset{\text{dot}}{\cdot} (\vec{v} \underset{\text{cross}}{\times} \vec{w})}$

as expected, a 3x3 matrix A is invertible iff $\det A \neq 0$

Method 1 given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$ find $\det A$

$\quad\ \ \vec{u}\ \ \vec{v}\ \ \vec{w}$

do cross product 1st:

$\begin{bmatrix} i & j & k \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}$

$\det A = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \cdot \left(\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \times \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \right)$

$$= \begin{bmatrix}1\\4\\7\end{bmatrix} \cdot \begin{bmatrix}2\\4\\-3\end{bmatrix}$$

$$= 2+16-21$$
$$= \boxed{-3}$$

$$\vec{v}\times\vec{w} = i \cdot \det\begin{bmatrix}5&8\\6&10\end{bmatrix} - j \cdot \det\begin{bmatrix}2&8\\3&10\end{bmatrix} + k \cdot \det\begin{bmatrix}2&5\\3&6\end{bmatrix}$$
$$= i(50-48) - j(20-24) + k(12-15)$$
$$= 2i \quad + 4j \quad - 3k$$

Method 2 Sarrus' Rule

① repeat 1st two column vectors outside matrix

given $A = \begin{bmatrix}1&2&3\\4&5&6\\7&8&10\end{bmatrix}\begin{matrix}1&2\\4&5\\7&8\end{matrix}$ find det A

$- \; - \; - \; + \; + \; +$

② construct 3 diagonals

③ =

④ multiply diagonals

$+ \; + \; +$

$\ominus \; \ominus \; \ominus$

$$\det A = (1)(5)(10) + 2(6)(7) + 3(4)(8) - 2(4)(10) - 1(6)(8) - 3(5)(7)$$
$$= 50 + 84 + 96 - 80 - 48 - 105$$
$$= 230 - 233 = \boxed{-3}$$

Conclusion: $\det A = -3 \neq 0$ $\therefore$ A is invertible

ex. find determinant of $A = \begin{bmatrix}a&b&c\\0&d&e\\0&0&f\end{bmatrix}$ upper triangular matrix

$$\begin{bmatrix}a&b&c\\0&d&e\\0&0&f\end{bmatrix}\begin{matrix}a&b\\0&d\\0&0\end{matrix}$$

$$\det A = adf + \cancel{be(0)}^{0} + \cancel{c(0)(0)}^{0} - \cancel{b(0)f}^{0} - \cancel{ae(0)}^{0} - \cancel{cd(0)}^{0}$$
$$\det A = adf$$

Conclusion: A is invertible iff adf is non-zero

<u>shortcut</u>: determinant of upper triangular matrix is product of entries of its diagonal

$$A = \begin{bmatrix}a&b&c\\0&d&e\\0&0&f\end{bmatrix} \quad \det A = \underline{adf}$$

ex. For which values of scalar λ is matrix A invertible?

given $A = \begin{bmatrix}\lambda&1&1\\1&\lambda&-1\\1&1&\lambda\end{bmatrix}\begin{matrix}\lambda&1\\1&\lambda\\1&1\end{matrix}$

$$\det A = \lambda^3 - \cancel{1} + \cancel{1} - \cancel{\lambda} + \cancel{\lambda} - \lambda \neq 0$$
$$\lambda^3 - \lambda \neq 0$$
$$\lambda(\lambda^2 - 1) \neq 0$$

$$\lambda^3 - \lambda \neq 0$$
$$\lambda(\lambda^2-1) \neq 0$$
$$\lambda(\lambda-1)(\lambda+1) \neq 0$$

conclusion: A is invertible as long as $\lambda \neq 0$, $\lambda \neq \pm 1$

ex. given $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, what is the relationship between $\det A$ and $\det B$ given $A = [\vec{u}\ \vec{v}\ \vec{w}]$

$B = [\vec{u}\ \vec{w}\ \vec{v}]$

$$\boxed{\text{Fact: cross product is anti-commutative i.e., } \vec{v}\times\vec{w} = -(\vec{w}\times\vec{v})}$$

$$\det A = \vec{u}\cdot(\vec{v}\times\vec{w})$$
$$\det B = \vec{u}\cdot(\vec{w}\times\vec{v})$$
$$\det B = -\vec{u}\cdot(\vec{v}\times\vec{w}) = -\det A$$

$$\boxed{\det B = -\det A}$$

ex. is $T: \mathbb{R}^3 \to \mathbb{R}$ a LT given $T\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \det\begin{bmatrix}2 & x_1 & 5\\3 & x_2 & 6\\4 & x_3 & 7\end{bmatrix}$

Do: $\det\begin{bmatrix}3 & x_1 & 5\\3 & x_2 & 6\\4 & x_3 & 7\end{bmatrix}\begin{matrix}2 & x_1\\3 & x_2\\4 & x_3\end{matrix}$ $= 14x_2 + 24x_1 + 15x_3 - 21x_1 - 12x_3 - 20x_2$

next, combine like terms

$= 3x_1 - 6x_2 + 3x_3$

conclusion: T is LT since output can be written as a LC

Determinant of 4×4 matrix

recall: determinant of upper-triangular matrix $\begin{bmatrix}a & b & c\\0 & d & e\\0 & 0 & f\end{bmatrix}$ is adf

ex. find $\det A$ given $A = \begin{bmatrix}1 & 3 & 1 & 4\\3 & 9 & 5 & 15\\0 & 2 & 1 & 1\\0 & 4 & 2 & 3\end{bmatrix}\begin{matrix}\\ \\ \text{III}\\ \text{IV}\end{matrix}$ technique: try to reduce A to be an upper triangular matrix

$$\begin{matrix}0 & -4 & -2 & -2\\0 & 4 & 2 & 3\\\hline 0 & 0 & 0 & 1\end{matrix}$$

$\downarrow$

swap rows $\begin{bmatrix}1 & 3 & 1 & 4\\3 & 9 & 5 & 15\\0 & 2 & 1 & 1\\0 & 0 & 0 & 1\end{bmatrix}$

note: negate determinant each time swap rows swap rows

$$\begin{bmatrix} 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

each time swap rows

$$-\begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 2 & 1 & 1 \\ 3 & 9 & 5 & 15 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \mathrm{I} \\ \\ \mathrm{III} \\ \\ \end{array}$$

$\mathrm{I} \rightarrow -3 \quad -9 \quad -3 \quad -12$

$\quad\quad 3 \quad 9 \quad 5 \quad 15$

$\quad\quad 0 \quad 0 \quad 2 \quad 3$

$$-\begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det A = -(1)(2)(2)(1)$$
$$= \boxed{-4}$$